ASYMMETRIC MODEL OF A VISCOUS ELECTROMAGNETIC FLUID

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Examples of media where account has to be taken of internal particle rotations, asymmetric features of the stress tensor, and the presence of torque stresses (see [1, 2]), are dielectric liquids and artifically synthesized ferromagnetic and ferroelectric liquids [3].

Ferromagnetic liquids [3] are homogeneous colloidal suspensions of ferromagnetic particles in a light liquid such as kerosene. The particles typically have a size of 25-100 Å, so that they are "single-domain" uniformily magnetized formations [3]. The concentration of such particles may reach 10^{18} cm^{-3}. Agglutination is prevented by Brownian movement and the presence of a dispersive agent (e. g., oleic acid) in the liquid. The physical mechanism of macroscopic magnetization of magnetic fluids by an external field is essentially linked with the fact that the particles have rotational degrees of freedom. Under the action of a magnetic field, a spatial electromagnetic force and spatial moment appear in the fluid and orient the particle magnetic dipoles along the field. The orienting action of the field is accompanied by field energy dissipation in work done by the spatial magnetic moment on the particle rotational displacements against the forces of viscous resistance to these rotations. During oriented rotation of particles located in a physically infinitesimal volume, changes occur in the magnetization vector per unit volume. Changes in the magnetization vector can also be produced in such liquids, e.g., by interaction of the particle rotations with hydrodynamic movement of their centers of inertia, thermal movement, or changes in the internal state of the particles themselves [3].

From what has just been said, complicated physico-mechanical processes must clearly occur when a magnetic liquid is subjected simultaneously to mechanical, electromagnetic, and thermal disturbances; these processes do in fact provide the explanation for the special macroscopic behavior of such liquids [3].

Before the behavior of magnetic and ferroelectric liquids can be described in macroscopic terms, it is essential to find models for continuous media with internal degrees of freedom [1], such that the influence of the average movement of the medium microstructures on its macroscopic behavior is taken into account. By providing supplementary degrees of freedom in such models, a variety of new interaction effects between mechanical, thermal, and electromagnetic phenomena may be explained.

A model of a ferromagnetic liquid was offered in [3], under a number of simplifying assumptions: notably, that the particles orient instantaneously along the field, and that movement of the medium does not influence the applied field. A nonisothermal model is described below for an isotropic, conducting, and dielectrically and paramagnetically polarizable liquid in which the particles perform interval rotations [2].

When magnetic media have been described in phenomenological terms in the literature, it has often been assumed (on the basis of the gyroscopic property of the magnetic moment) that the intensity of magnetization vector is proportional to the internal macroscopic moment of momentum density vector [4]. An initial hypothesis of this type is justified when describing media in which the magnetic moment is determined, e.g., by the molecular internal-rotation vector, the electron spins, or orbital electron movement, etc. When suspensions containing magnetic particles are considered [3], the internal-rotation vector describes the rotation of the particles as they orient their moments along the field. In view of this, it cannot be assumed for a suspension that the intensity of magnetization vector is proportional to the particle internal rotation vector. When developing the present model, the methods of thermodynamics of irreversible

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processes are used; with these methods, the electromagnetic field energy dissipation on relaxation processes linked with polarization and magnetization phenomena in the medium can be taken into account under extremely simple and general assumptions [12]. The field energy dissipation is taken into account by considering the invariant time derivatives of the polarization and magnetization vectors as the definitive parameters [5].

The paper discusses the form of the dependence of the free energy function on the independent definitive parameters, the number and form of which are determined by means of the axiom of equal representability, the axiom of objectivity [6], and the Clausius–Duhem local inequality. When obtaining the complete system of definitive linear equations in the thermodynamic forces, the possible dependence of the phenomenological coefficients on the magnetic field is ignored. If such a dependence is taken into account in the definitive electromechanical equations, hydrodynamic generalizations of the Bloch–Blombergen [7] and Voigt [8] equations are obtained.

Our model is characterized by various kinds of cross effects between mechanical, thermal and electromagnetic phenomena. In one of the limiting cases, where there is no interaction between the continuous medium and the electromagnetic field, the model is the same as the nonisothermal Grade model [2]. In another limiting case, where dissipative phenomena, linked with intensity of magnetization and polarization of the medium and with internal rotation of the particles, may be ignored, the model is the same as in [3].

1. The Fundamental Equations of Electrodynamics, Mechanics, and Thermodynamics. Chu's form [9] of the equations of electrodynamics of a moving medium (polarization model) in the international system of units is

$$\begin{gathered}\nabla\times\mathbf{h}-\frac{\partial\varepsilon_0\mathbf{e}}{\partial t}=\frac{\partial\mathbf{p}}{\partial t}+\nabla\times(\mathbf{p}\times\mathbf{v})+\mathbf{i},\quad \nabla\cdot\mu_0\mathbf{h}=-\nabla\cdot\mu_0\mathbf{m}\\ \nabla\times\mathbf{e}+\frac{\partial\mu_0\mathbf{h}}{\partial t}=-\frac{\partial\mu_0\mathbf{m}}{\partial t}-\nabla\times(\mu_0\mathbf{m}\times\mathbf{v}),\quad \nabla\cdot\varepsilon_0\mathbf{e}=-\nabla\cdot\mathbf{p}\end{gathered}\tag{1.1}$$

where e and h are the electric and magnetic field strength vectors, p and m are the polarization and intensity of magnetization vectors, i is the electric current density vector, ε_0 and μ_0 are the dielectric constant and permeability in vacuo, ∇ is the Hamilton operator [10], and $\times$ denotes vector multiplication. It should be noted that these equations hold for media whose velocities are substantially less than the velocity of light in vacuo $c=(\varepsilon_0\mu_0)^{-1/2}$.

The electromagnetic field momentum and energy densities are

$$\mathbf{g}=\mathbf{S}/c^2,\quad w=\tfrac{1}{2}(\varepsilon_0\mathbf{e}\cdot\mathbf{e}+\mu_0\mathbf{h}\cdot\mathbf{h}),\quad \mathbf{S}=\mathbf{e}\times\mathbf{h}\tag{1.2}$$

where S is the Poynting vector.

Using (1.2) in conjunction with (1.1), the laws of conservation of the electromagnetic field momentum and energy densities may be obtained in a form convenient for the thermodynamic description of polarized and magnetized media:

$$\begin{gathered}\frac{\partial\mathbf{g}}{\partial t}=\nabla\cdot\boldsymbol{\tau}-\boldsymbol{\varphi},\quad \frac{\partial w}{\partial t}=\nabla\cdot(\boldsymbol{\tau}_1\cdot\mathbf{v}-\mathbf{S})-\Pi\\ \boldsymbol{\tau}=\boldsymbol{\tau}_0+\boldsymbol{\tau}_1,\quad \boldsymbol{\tau}_0=\varepsilon_0\mathbf{ee}+\mu_0\mathbf{hh}-w\mathbf{I},\quad \boldsymbol{\tau}_1=\mathbf{m}\boldsymbol{\eta}+\mathbf{p}\boldsymbol{\varepsilon}\\ \boldsymbol{\varphi}=\mathbf{i}\times\mu_0\mathbf{h}+\mathbf{p}\cdot\nabla\mathbf{e}+\mu_0\mathbf{m}\cdot\nabla\mathbf{h}+\rho\boldsymbol{\pi}^{\cdot}\times\mu_0\mathbf{h}-\rho\mu_0\boldsymbol{\mu}^{\cdot}\times\varepsilon_0\mathbf{e}\\ +\mathbf{v}\times(\mathbf{p}\cdot\nabla)\mu_0\mathbf{h}-\mathbf{v}\times(\mu_0\mathbf{m}\cdot\nabla)\varepsilon_0\mathbf{e},\quad \boldsymbol{\pi}=\mathbf{p}/\rho,\quad \boldsymbol{\mu}=\mathbf{m}/\rho\\ \Pi=\mathbf{i}\cdot\mathbf{e}+\rho\boldsymbol{\pi}^{\cdot}\cdot\mathbf{e}+\rho\mu_0\boldsymbol{\mu}^{\cdot}\cdot\mathbf{h}+\mathbf{v}\cdot(\mathbf{p}\cdot\nabla\mathbf{e}-\mu_0\mathbf{m}\cdot\nabla\mathbf{h})\\ \rho\boldsymbol{\pi}^{\cdot}=\mathbf{dp}/\partial t+\nabla\cdot(\mathbf{v}\times\mathbf{p}),\quad \boldsymbol{\varepsilon}=\mathbf{e}+\mathbf{v}\times\mu_0\mathbf{h},\quad \boldsymbol{\eta}=\mathbf{h}-\mathbf{v}\times\varepsilon_0\mathbf{e}\end{gathered}\tag{1.3}$$

Here, ρ is the mass density, π and μ are the electric and magnetic polarizations per unit mass, τ is the electromagnetic tension tensor, which, to facilitate thermodynamic description of the polarization medium, is written as the sum of the vacuum tension tensor τ_0 and the material tension tensor τ_1, φ is the ponderomotive force of interaction between the electromagnetic field and the material medium, Π is the density of electrodynamic power consumption, linked with the conductivity and the electric and magnetic polarizations of the medium, ε and η are the electric and magnetic effective field strength vectors [9], and I is the unit dyad. The dot operation (·) denotes the total time derivative.

Notice that Eq. (1.3) for the pondermotive force is derived in [11], where the expression for the force density is obtained by starting from the general principle of virtual displacements. An interpretation of the various terms in Eq. (1.3) may be found in [11].

A structured material continuum will be considered, at each point of which we find the translation velocity vector $\mathbf{v}$ and angular velocity vector ω of the particles from which the point is made up. It is assumed that a mechanical force with vector f and arbitrary material volume V is applied to each point, while the surface S of V is acted on by the force stress vector $\mathbf{t}_n$, the momentum stress vector $\mathbf{s}_n$, and the electromagnetic tension vector τ_n. The equation of mass conservation, the equations for the variations of momentum and moment of momentum, the equation of energy variation, and the Clausius–Duhem inequality for the entropy production [6], are written in the integral form

$$\frac{d}{dt}\int_V \rho\, dV = 0, \quad \frac{d}{dt}\int_V (\rho\mathbf{v}+\mathbf{g})\, dV = \int_S (\boldsymbol{\tau}_n + \mathbf{t}_n + v_n\mathbf{g})\, dS + \int_V \rho\mathbf{f}\, dV$$
$$\frac{d}{dt}\int_V [(\mathbf{r}\times\mathbf{v} + J\boldsymbol{\omega})\rho + \mathbf{r}\times\mathbf{g}]\, dV = \int_S [\mathbf{r}\times(\boldsymbol{\tau}_n + \mathbf{t}_n + v_n\mathbf{g}) + \mathbf{s}_n]\, dS + \int_V \mathbf{r}\times\rho\mathbf{f}\, dV$$
$$\frac{d}{dt}\int_V \left[\left(\frac{\mathbf{v}^2}{2} + u + J\frac{\boldsymbol{\omega}^2}{2}\right)\rho + w\right] dV = \int_S [(\mathbf{t}_n + \boldsymbol{\tau}_n + v_n\mathbf{g})\cdot\mathbf{v} + \mathbf{s}_n\cdot\boldsymbol{\omega}]\, dS \tag{1.4}$$
$$-\int_S \mathbf{n}\cdot(\mathbf{q}+\mathrm{S}^\circ)\, dS + \int_V (\rho\mathbf{f}\cdot\mathbf{v} + r)\, dV \quad \int_V \rho\gamma\, dV \geq \frac{d}{dt}\int_V \rho\eta\, dV - \int_V \rho\frac{r}{T}\, dV + \int_V \frac{\mathbf{q}}{T}\cdot\mathbf{n}\, dS \geqslant 0$$

Here, $\mathbf{r}$ is the position vector of the point in the material continuum, $\mathbf{q}$ is the heat flux vector, r the scalar density of the heat source per unit mass and unit time [6], u the internal energy density per unit mass, $\mathrm{S}^\circ = \boldsymbol{\varepsilon}\times\eta$ the electromagnetic energy flux vector through the moving surface S, J the mean moment of inertia per unit mass, $\mathbf{n}$ the outward normal to the surface S, v_n the velocity projection on the normal $\mathbf{n}$, η the entropy density per unit mass, γ the entropy production density per unit mass, and T the temperature. When writing the equation of energy variation in (1.4), it is assumed that the moment stresses only do work on the internal rotational displacements [2].

Use will be made of the following relationships, connecting the moment stress dyad $\mathbf{s}$, the mechanical stress dyad $\mathbf{t}$, and the electromagnetic tension dyad τ, with the moment stress vector $\mathbf{s}_n$, the mechanical stress vector $\mathbf{t}_n$, and the electromagnetic tension vector τ_n [2, 13]:

$$\mathbf{t}_n = \mathbf{n}\cdot\mathbf{t}, \quad \mathbf{s}_n = \mathbf{n}\cdot\mathbf{s}, \quad \boldsymbol{\tau}_n = \mathbf{n}\cdot\boldsymbol{\tau} \tag{1.5}$$

Recalling the expressions of dyadic calculus [10] and connecting integration over the surface S with integration over the continuous volume V, the following are obtained from Eqs. (1.4), (1.3), and (1.5):

$$\rho^\cdot + \rho\nabla\cdot\mathbf{v} = 0, \ \rho\mathbf{v}^\cdot = \nabla\cdot\mathbf{t} + \rho\mathbf{f} + \boldsymbol{\varphi}, \quad \nabla\cdot\mathbf{s} + \boldsymbol{\tau}\times\mathbf{I} + \mathbf{t}\times\mathbf{I} = \rho J\boldsymbol{\omega}^\cdot$$
$$\rho u^\cdot - (\rho r - \nabla\cdot\mathbf{q}) = \mathbf{t}\cdot\cdot\nabla\mathbf{v} + \mathbf{s}\cdot\cdot\nabla\boldsymbol{\omega} - \mathbf{t}\times\cdot\mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{i}\cdot\boldsymbol{\varepsilon} + \rho\mu_0\boldsymbol{\mu}^*\cdot\boldsymbol{\eta} + \rho\boldsymbol{\pi}^*\cdot\boldsymbol{\varepsilon} \tag{1.6}$$
$$\rho T\gamma = \rho T\eta^\cdot - (\rho r - \nabla\cdot\mathbf{q}) - T^{-1}\mathbf{q}\cdot\nabla T \geqslant 0, \ \rho\boldsymbol{\pi}^* = \rho\boldsymbol{\pi}^\cdot - \boldsymbol{\omega}\times\mathbf{p}, \ \rho\boldsymbol{\mu}^* = \rho\boldsymbol{\mu}^\cdot - \boldsymbol{\omega}\times\mathbf{m}$$

where $(\times\cdot)$ denotes that vector multiplication is performed on the left, and scalar multiplication on the right factors of the dyads, and $(\cdot\,\cdot)$ means that the left and right factors of the dyads are both multiplied scalarly.

Introducing the specific free-energy function

$$\varphi = u - T\eta \tag{1.7}$$

the local Clausius–Duhem inequality and the equations of energy variation in (1.6) may be written as

$$\rho T\gamma = \mathbf{t}\cdot\cdot\nabla\mathbf{v} + \mathbf{s}\cdot\cdot\nabla\boldsymbol{\omega} - \mathbf{t}\times\mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{i}\cdot\boldsymbol{\varepsilon} + \rho\mu_0\boldsymbol{\mu}^*\cdot\boldsymbol{\eta} + \rho\boldsymbol{\pi}^*\cdot\boldsymbol{\varepsilon} - \rho(\varphi^\cdot - \eta T) - \frac{1}{T}\mathbf{q}\cdot\nabla T \geqslant 0 \tag{1.8}$$

2. The Linear Definitive Equations. The definitive equations will be considered for an isotropic liquid with internal particle rotations when the phenomena of heat conductivity, electrical conductivity, and dielectric and paramagnetic relaxation are present. It will be assumed that the independent definitive parameters for the liquid are

$$\rho^{-1}, T, \mathbf{v}, \boldsymbol{\omega}, \mathbf{p}, \mathbf{m}, \nabla T, \nabla\mathbf{v}, \nabla\boldsymbol{\omega}, \rho\boldsymbol{\pi}^*, \rho\boldsymbol{\mu}^* \tag{2.1}$$

Here, the gradients of the polarization and intensity of magnetization vectors are not regarded as definitive arguments. This means that inhomogeneities in the magnetization and polarization of the medium are ignored in our model [5, 13]. It is possible to make this assumption in view of the uniform distribution [3, 13] of ferromagnetic "single-domain" homogeneously magnetized particles in a liquid carrier. By taking the time derivatives of the polarization and magnetization vectors as independent definitive parameters, the phenomena of electric and magnetic relaxation in the medium may be described [5]. It can be seen immediately from the form of the time derivatives in (1.6) that variation of the polarization and intensity of magnetization vectors due to particle rotations is here taken into account.

On recalling the requirements of the axiom of objectivity [6], it can be shown that the arguments (2.1) have the objective forms

$$\rho^{-1},\ T,\ \mathbf{p},\ \mathbf{m},\ \nabla\times\mathbf{v}-2\boldsymbol{\omega},\ \nabla T,\ (\nabla\mathbf{v})^{s},\ \nabla\boldsymbol{\omega},\ \rho\boldsymbol{\pi}^{*},\ \rho\boldsymbol{\mu}^{*} \tag{2.2}$$

where the superscript s denotes the symmetric part of the dyad. When obtaining these forms, the antisymmetric dyad $(\nabla\mathbf{v})^{a}$ was replaced by its equivalent vector $\mathbf{c}$. The connections between these latter are

$$(\nabla\mathbf{v})^{a}=-\mathbf{I}\times\mathbf{c},\quad \mathbf{c}=\tfrac{1}{2}(\nabla\mathbf{v})\times\cdot\mathbf{I}=\tfrac{1}{2}\nabla\times\mathbf{v} \tag{2.3}$$

The dependent thermomechanical and electromagnetic definitive variables for our model of the liquid are

$$\mathbf{t},\ \mathbf{s},\ \mathbf{q},\ \boldsymbol{\eta},\ \varphi,\ \varepsilon,\ \eta,\ \mathbf{i} \tag{2.4}$$

According to the axiom of equal representability [6], all the dependent definitive variables (2.4) must be functions of the same set of independent definitive arguments (2.2), while the converse is not proved. Consequently, the free energy function is

$$\varphi=\varphi(\rho^{-1},\ T,\ \mathbf{p},\ \mathbf{m},\ \rho\boldsymbol{\mu}^{*},\ \rho\boldsymbol{\pi}^{*},\ \nabla\times\mathbf{v}-2\boldsymbol{\omega},\ \nabla T,\ (\nabla\mathbf{v})^{s},\ \nabla\boldsymbol{\omega}) \tag{2.5}$$

The Clausius–Duhem local inequality (1.8) imposes certain restrictions on the form of the free energy function (2.5). Once these have been found, the dependent definitive variables $\mathbf{t}$, $\mathbf{s}$, ε, η, $\mathbf{q}$, and $\mathbf{i}$ can be obtained in the linear approximation by the methods of thermodynamics of irreversible processes [12]. In fact, from Eqs. (1.8) and (2.5), regarding the free energy φ as a differentiable function of its arguments, and recalling the equation of continuity (1.6), we have

$$\begin{aligned}\rho T\gamma=\rho\Big(\eta-\frac{\partial\varphi}{\partial T}\Big)\frac{dT}{dt}+\Big[\mathbf{t}+\Big(\rho\frac{\partial\varphi}{\partial\mathbf{p}}\cdot\mathbf{p}+\rho\frac{\partial\varphi}{\partial\mathbf{m}}\cdot\mathbf{m}-\frac{\partial\varphi}{\partial\rho^{-1}}\Big)\mathbf{I}\Big]\cdot\cdot\,\nabla\mathbf{v}-\frac{1}{T}\mathbf{q}\cdot\nabla T\\ +\mathbf{s}\cdot\cdot\,\nabla\boldsymbol{\omega}-\mathbf{t}\times\mathbf{I}\cdot\boldsymbol{\omega}+\mathbf{i}\cdot\boldsymbol{\varepsilon}+\rho\mu_{0}\boldsymbol{\mu}^{*}\cdot\boldsymbol{\eta}+\rho\boldsymbol{\pi}^{*}\cdot\boldsymbol{\varepsilon}-\rho^{2}\frac{\partial\varphi}{\partial\mathbf{p}}\cdot\boldsymbol{\pi}^{\cdot}-\rho^{2}\frac{\partial\varphi}{\partial\mathbf{m}}\cdot\boldsymbol{\mu}^{\cdot}\\ -\rho\frac{\partial\varphi}{\partial(\nabla\mathbf{v})^{s}}\cdot\cdot\,\frac{d(\nabla\mathbf{v})^{s}}{dt}-\rho\frac{\partial\varphi}{\partial(\nabla\times\mathbf{v}-2\boldsymbol{\omega})}\cdot\frac{d(\nabla\times\mathbf{v}-2\boldsymbol{\omega})}{dt}-\rho\frac{\partial\varphi}{\partial(\nabla\boldsymbol{\omega})}\cdot\cdot\,\frac{d(\nabla\boldsymbol{\omega})}{dt}\\ -\rho\frac{\partial\varphi}{\partial(\nabla T)}\cdot\frac{d(\nabla T)}{dt}-\rho\frac{\partial\varphi}{\partial\rho\boldsymbol{\pi}^{*}}\cdot\frac{d\rho\boldsymbol{\pi}^{*}}{dt}-\rho\frac{\partial\varphi}{\partial\rho\boldsymbol{\mu}^{*}}\cdot\frac{d\rho\boldsymbol{\mu}^{*}}{dt}\geqslant 0\end{aligned} \tag{2.6}$$

Since, $\mathbf{t}$, $\mathbf{s}$, $\mathbf{q}$, $\boldsymbol{\eta}$, ε, and $\mathbf{i}$ are independent of the material time derivatives of the quantities T, ΔT, $\nabla\times\mathbf{v}-2\boldsymbol{\omega}$, $\rho\boldsymbol{\pi}^{*}$, $\rho\boldsymbol{\mu}^{*}$, $(\nabla\mathbf{v})^{S}$, and $\nabla\boldsymbol{\omega}$, and noting that (2.6) is linear in these derivatives, the necessary and sufficient conditions for Eq. (2.6) to hold for any independent variations of these derivatives are

$$\begin{aligned}\rho T\gamma=\Big[\mathbf{t}+\Big(\rho\frac{\partial\varphi}{\partial\mathbf{p}}\cdot\mathbf{p}+\rho\frac{\partial\varphi}{\partial\mathbf{m}}\cdot\mathbf{m}-\frac{\partial\varphi}{\partial\rho^{-1}}\Big)\mathbf{I}\Big]\cdot\cdot\,\nabla\mathbf{v}-\rho^{2}\frac{\partial\varphi}{\partial\mathbf{p}}\cdot\boldsymbol{\pi}^{\cdot}-\rho^{2}\frac{\partial\varphi}{\partial\mathbf{m}}\cdot\boldsymbol{\mu}^{\cdot}\\ +\mathbf{s}\cdot\cdot\,\nabla\boldsymbol{\omega}-\mathbf{t}\times\mathbf{I}\cdot\boldsymbol{\omega}+\mathbf{i}\cdot\boldsymbol{\varepsilon}+\rho\mu_{0}\boldsymbol{\mu}^{*}\cdot\boldsymbol{\eta}+\rho\boldsymbol{\pi}^{*}\cdot\boldsymbol{\varepsilon}-\frac{1}{T}\mathbf{q}\cdot\nabla T\geqslant 0\\ \frac{\partial\varphi}{\partial(\nabla\mathbf{v})^{S}}=\frac{\partial\varphi}{\partial(\nabla\times\mathbf{v}-2\boldsymbol{\omega})}=\frac{\partial\varphi}{\partial(\nabla\boldsymbol{\omega})}=\frac{\partial\varphi}{\partial(\nabla T)}=\frac{\partial\varphi}{\partial\rho\boldsymbol{\pi}^{*}}=\frac{\partial\varphi}{\partial\rho\boldsymbol{\mu}^{*}}=\eta-\frac{\partial\varphi}{\partial T}=0\end{aligned} \tag{2.7}$$

All in all, the free energy function can be written as

$$\varphi=\varphi(\rho^{-1},\ T,\ \mathbf{p},\ \mathbf{m}),\quad \eta=\partial\varphi/\partial T \tag{2.8}$$

At the same time, according to the axiom of objectivity [6], the free energy function φ must be form-invariant under movement of the spatial reference system as a rigid whole. Recalling Cauchy's theorem (see [14], p. 144) and Curie's theorem [12], it can be shown that φ must in this case be a function of the independent scalar invariants

$$\rho^{-1},\ T,\ J_1 = \mathbf{p}\cdot\mathbf{p},\quad J_2 = \mathbf{m}\cdot\mathbf{m} \tag{2.9}$$

In view of Eq. (2.9) and the identity $\boldsymbol{a}\cdot(\boldsymbol{a}\times\mathbf{b}) = 0$, the Clausius–Duhem local inequality (2.7) can be written as

$$\rho T\gamma = (\mathbf{t} + p\mathbf{I})\cdot\cdot\nabla\mathbf{v} + \mathbf{s}\cdot\cdot\nabla\boldsymbol{\omega} - \mathbf{t}\times\cdot\mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{i}\cdot\boldsymbol{\varepsilon} + \rho\mu_0\boldsymbol{\mu}^*\cdot(\boldsymbol{\eta} - {}_E\boldsymbol{\eta}) + \rho\boldsymbol{\pi}^*\cdot(\boldsymbol{\varepsilon} - {}_E\boldsymbol{\varepsilon}) - \frac{1}{T}\mathbf{q}\cdot\nabla T \geqslant 0 \tag{2.10}$$

where the following notation is used:

$$p = -\frac{\partial\varphi}{\partial\rho^{-1}} + 2\rho\frac{\partial\varphi}{\partial J_1}J_1 + 2\rho\frac{\partial\varphi}{\partial J_2}J_2,\quad {}_E\boldsymbol{\varepsilon} = 2\rho\frac{\partial\varphi}{\partial J_1}\mathbf{p},\quad {}_E\boldsymbol{\eta} = \frac{2\rho}{\mu_0}\frac{\partial\psi}{\partial J_2}\mathbf{m} \tag{2.11}$$

and p is the hydrostatic pressure in the polarized and magnetized liquid, E^{ε} and E^{η} are the local electric and local magnetic field strength vectors [8], and $\boldsymbol{\varepsilon} - \mathrm{E}^{\varepsilon}$ and $\boldsymbol{\eta} - \mathrm{E}^{\eta}$ are the dissipative parts of the electric and magnetic field strength vectors, characterizing the relaxation processes in magnetization and polarization.

To facilitate isolation of the independent forces and thermodynamic fluxes, the dyads $\mathbf{t}$, $\mathbf{s}$, $\nabla\mathbf{v}$, and $\nabla\boldsymbol{\omega}$ will be written as

$$\begin{gathered}\mathbf{t} = -p\mathbf{I} + \boldsymbol{\beta} = (\beta^\circ - p)\mathbf{I} + \boldsymbol{\beta}^d + \boldsymbol{\beta}^a,\quad \mathbf{s} = s^\circ\mathbf{I} + \mathbf{s}^a + \mathbf{s}^d,\quad \beta^\circ = {}^1/_3\boldsymbol{\beta}\cdot\cdot\mathbf{I}\\ \nabla\mathbf{v} = {}^1/_3\nabla\cdot\mathbf{v}\mathbf{I} + (\nabla\mathbf{v})^a + (\nabla\mathbf{v})^d,\quad \nabla\boldsymbol{\omega} = {}^1/_3\nabla\cdot\boldsymbol{\omega}\mathbf{I} + (\nabla\boldsymbol{\omega})^a + (\nabla\boldsymbol{\omega})^d,\\ s^\circ = {}^1/_3\,\mathbf{s}\cdot\cdot\mathbf{I}\end{gathered} \tag{2.12}$$

where the superscript a refers to antisymmetric dyads and d to the deviator parts of the symmetric dyads.

Using (2.12), the inequality (2.10) for the production of entropy can be written as

$$\begin{gathered}-\frac{1}{T}\mathbf{q}\cdot\nabla T + \beta^\circ\nabla\cdot\mathbf{v} + s^\circ\nabla\cdot\boldsymbol{\omega} + \mathbf{P}^a\cdot(\nabla\times\mathbf{v} - 2\boldsymbol{\omega}) + \mathbf{d}\cdot 2\mathbf{b} + \mathbf{s}^d\cdot\cdot(\nabla\boldsymbol{\omega})^a\\ + \boldsymbol{\beta}^d\cdot\cdot(\nabla\mathbf{v})^d + \mathbf{i}\cdot\boldsymbol{\varepsilon} + \rho\mu_0\boldsymbol{\mu}^*\cdot(\boldsymbol{\eta} - {}_E\boldsymbol{\eta}) + \rho\boldsymbol{\pi}^*(\boldsymbol{\varepsilon} - {}_E\boldsymbol{\varepsilon}) \geqslant 0\end{gathered} \tag{2.13}$$

When obtaining (2.13), the dyads $\boldsymbol{\beta}^a$ and $(\nabla\mathbf{v})^a$ and pseudodyads $(\nabla\boldsymbol{\omega})^a$ and $\mathbf{s}^a$ were replaced (see [2]) by their equivalent pseudovectors $\mathbf{P}^a$, $1/2\nabla\times\mathbf{v}$ and polar vectors $\mathbf{b}$ and $\mathbf{d}$. The connections between these quantities are similar to (2.3).

Notice that, in (2.13), $\nabla\cdot\mathbf{v}$ is a scalar, $(\nabla\mathbf{v})^{\mathrm{d}}$ is a dyad, $\nabla\cdot\boldsymbol{\omega}$ is a pseudoscalar, and $(\nabla\boldsymbol{\omega})^{\mathrm{d}}$ a pseudodyad. The quantities $\nabla\mathrm{T}, \boldsymbol{\varepsilon}, \rho\boldsymbol{\pi}^*$ are polar vectors, and $\nabla\times\mathbf{v} - 2\boldsymbol{\omega}$, $\rho\mu_0\boldsymbol{\mu}^*$ are pseudovectors [13, 15]. The thermodynamic forces $\nabla\mathrm{T}/\mathrm{T}$, $\boldsymbol{\varepsilon}$, $\rho\mu_0\boldsymbol{\mu}^*$ in (2.13) are even functions of time t, while $2\mathbf{b}$, $\rho\boldsymbol{\pi}^*$, $\nabla\times\mathbf{v} - 2\boldsymbol{\omega}$ are odd functions [12, 13].

Recalling Curie's theorem and Onsager's interaction relationship [12], the complete system of phenomenological equations for scalar and pseudoscalar phenomena, linear in the thermodynamic forces, may be obtained:

$$\beta^\circ = \alpha_1\nabla\cdot\mathbf{v},\qquad s^\circ = \gamma_1\nabla\cdot\boldsymbol{\omega} \tag{2.14}$$

for dyadic and pseudodyadic phenomena

$$\boldsymbol{\beta}^d = 2\alpha_2(\nabla\mathbf{v})^d,\quad \mathbf{s}^d = 2\gamma_2(\nabla\boldsymbol{\omega})^d \tag{2.15}$$

and for pseudovector and vector phenomena

$$\begin{gathered}\boldsymbol{\eta} - {}_E\boldsymbol{\eta} = h_1\rho\mu_0\boldsymbol{\mu}^* - \alpha_4(\nabla\times\mathbf{v} - 2\boldsymbol{\omega})\\ \mathbf{P}^a = \alpha_3(\nabla\times\mathbf{v} - 2\boldsymbol{\omega}) + \alpha_4\rho\mu_0\boldsymbol{\mu}^*\\ \mathbf{d} = 2\gamma_3\mathbf{b} + \varkappa_2\nabla T + \gamma_4\rho\boldsymbol{\pi}^* + \gamma_5\boldsymbol{\varepsilon}\\ \mathbf{q} = -\varkappa_1\nabla T + 2\varkappa_2 T\mathbf{b} + \varkappa_3 T\rho\boldsymbol{\pi}^* - \varkappa_4 T\boldsymbol{\varepsilon}\\ \mathbf{i} = \sigma_1\boldsymbol{\varepsilon} - l_2\rho\boldsymbol{\pi}^* + \varkappa_4\nabla T - 2\gamma_5\mathbf{b}\\ \boldsymbol{\varepsilon} - {}_E\boldsymbol{\varepsilon} = l_1\rho\boldsymbol{\pi}^* + \varkappa_3\nabla T + 2\gamma_4\mathbf{b} + l_2\boldsymbol{\varepsilon}\end{gathered} \tag{2.16}$$

In Eqs. (2.14)–(2.16), the phenomenological coefficients are scalars, characterizing the isotropic properties of the medium. It may be noted that they are, in general, functions of the density and temperature.

It is immediately evident from (2.16) that our model is characterized by thermomechanical, thermoelectric, thermopolarization, electric-polarization, electromechanical, polarization-mechanical, and

magnetomechanical cross effects. The sizes of the contributions of these cross effects to the thermodynamic fluxes are determined respectively by the phenomenological coefficients $2\varkappa_2$, $\varkappa_4$, $\varkappa_3$, l_2, γ_5, α_4. Notice, for example, that in the context of this model a displacement of the liquid will be accompanied by the appearance of magnetization and polarization of the liquid, heat flux, and a mechanico-electrical current.

The definitive equations (2.16) for the dissipative parts of the electric and magnetic field strength vectors are equations of molecular equilibrium [8], in which allowance is made for dissapative forces of resistance to polarization and magnetization effects and for intersection of thermodynamic fluxes. These equations can also be regarded as describing the time variations of the polarization and magnetization vectors, due to convective moment of the medium, internal rotations, the process of electric and magnetic relaxation, and cross effects.

When obtaining Eqs. (2.14)-(2.16), no account was taken of possible dependence of the phenomenological coefficients on the magnetic field [7, 12]. If such a dependence is brought in, nonlinear terms in the field strengths and thermodynamic forces appear in (2.14)-(2.16). Such a dependence will only be considered here for the coeffients of the electromechanical definitive equations (2.16). From purely practical considerations, the quantity $\eta - E^{\eta}$ will be regarded as an independent thermodynamic force. Disregarding the intersections of the thermodynamic fluxes $\varepsilon - E^{\varepsilon}$ and $\rho\mu_0\mu^*$ with the other fluxes, the following linear definitive equations are obtained for them from (2.13):

$$\varepsilon - {}_E\varepsilon = \mathbf{L}\cdot\rho\pi^*, \qquad \rho\mu_0\mu^* = \mathbf{R}\cdot(\eta - {}_E\eta) \tag{2.17}$$

Here, the symmetric parts $\mathbf{L}^S$ and $\mathbf{R}^S$, and antisymmetric parts $\mathbf{L}^a$ and $\mathbf{R}^a$, of the dyads of phenomenological coefficients $\mathbf{L}$ and $\mathbf{R}$ satisfy the Onsager interaction relationships [7, 13]

$$\begin{aligned} \mathbf{L}^s(\eta) &= \mathbf{L}^s(-\eta), & \mathbf{R}^s(\eta) &= \mathbf{R}^s(-\eta) \\ \mathbf{l}^a(\eta) &= -\mathbf{l}^a(-\eta), & \mathbf{r}^a(\eta) &= -\mathbf{r}^a(-\eta) \end{aligned} \tag{2.18}$$

When obtaining these expressions, the antisymmetric dyads $\mathbf{L}^a$ and $\mathbf{R}^a$ were replaced by their equivalent pseudovectors $\mathbf{l}^a$ and $\mathbf{r}^a$. It is clear from (2.18) that the symmetric parts of the dyads of phenomenological coefficeints are even functions of the magnetic field, while the pseudovectors are odd functions.

For an isotropic liquid, Eq. (2.18) will be satisfied in the linear approximation in η by putting

$$\mathbf{L} = l_0\mathbf{I} - \mathbf{I}\times\lambda_1\eta, \qquad \mathbf{R} = r_1\mathbf{I} - \mathbf{I}\times r_2\eta \tag{2.19}$$

Using (2.19) and (2.17),

$$\varepsilon - {}_E\varepsilon = l_1\rho\pi^* + \lambda_1\rho\pi^*\times\eta, \quad \rho\mu_0\mu^* = r_1(\eta - {}_E\eta) - r_{2E}\times\eta \tag{2.20}$$

This last shows that, when the coefficients are regarded as dependent on the magnetic field, the definitive equations acquire extra terms which take account of the change in the polarization and magnetization vectors, due to the gyrotropic properties of the medium [13] with scalar coefficient of gyrotropy λ_1 and to the gyroscopic property of the magnetic moment [13], with magnetomechanical ratio r_2.

Using Eqs. (2.3), (2.12), and (2.14)-(2.16), the equation of continuity, the equation of motion (1.6), and the inequality (2.10) for the entropy production can be written as

$$\begin{aligned} \rho\mathbf{v}^{\cdot} = &-\nabla p + \nabla(\alpha_1\nabla\cdot\mathbf{v}) + 2\nabla\cdot[\alpha_2(\nabla\mathbf{v})^d] + \nabla\times[\alpha_3(2\boldsymbol{\omega} - \nabla\times\mathbf{v})] \\ &- \nabla\times(\alpha_4\rho\mu_0\mu^*) + \rho\mathbf{f} + \boldsymbol{\varphi}, \qquad \rho^{\cdot} + \rho\nabla\cdot\mathbf{v} = 0 \\ \rho J\boldsymbol{\omega}^{\cdot} = &\nabla(\gamma_1\nabla\cdot\boldsymbol{\omega}) + 2\nabla\cdot[\gamma_2(\nabla\boldsymbol{\omega})^d] + 2\nabla\cdot[\gamma_3(\nabla\boldsymbol{\omega})^a] - \nabla\cdot(\varkappa_2\mathbf{I}\times\nabla T) \\ &- \nabla\cdot(\gamma_4\rho\mathbf{I}\times\boldsymbol{\pi}^*) - \nabla\cdot(\gamma_5\mathbf{I}\times\boldsymbol{\varepsilon}) + 2\alpha_3(\nabla\times\mathbf{v} - 2\boldsymbol{\omega}) + 2\alpha_4\rho\mu_0\mu^* + \mathbf{p}\times\boldsymbol{\varepsilon} + \mu_0\mathbf{m}\times\boldsymbol{\eta} \\ \rho\gamma = &\frac{1}{T}\varkappa_1(\nabla T)^2 + \alpha_1(\nabla\cdot\mathbf{v})^2 + \gamma_1(\nabla\cdot\boldsymbol{\omega})^2 + 4\gamma_3\mathbf{b}^2 + h_1(\rho\mu_0\mu^*)^2 + l_1(\rho\pi^*)^2 \\ &+ \sigma_1\varepsilon^2 + \alpha_3(\nabla\times\mathbf{v} - 2\boldsymbol{\omega})^2 + 2\varkappa_4\boldsymbol{\varepsilon}\cdot\nabla T + 4\gamma_4\rho\boldsymbol{\pi}^*\cdot\mathbf{b} + 2\alpha_2(\nabla\mathbf{v})^d\cdot\cdot(\nabla\mathbf{v})^d + 2\gamma_2(\nabla\boldsymbol{\omega})^d\cdot\cdot(\nabla\boldsymbol{\omega})^a \end{aligned} \tag{2.21}$$

Since Eqs. (2.21) for the entropy production, which is quadratic in the thermodynamic forces, must be positive, the following restrictions are obtained on the phenomenological coefficients:

$$\alpha_1,\ \gamma_1,\ \alpha_2,\ \gamma_2,\ h_1,\ \alpha_3,\ \gamma_3,\ \varkappa_1,\ \sigma_1,\ l_1 \geqslant 0,\ \varkappa_4^2 \leqslant \sigma_1\varkappa_1,\ 2\gamma_4^2 \leqslant l_1\gamma_3 \tag{2.22}$$

The terms with coefficients α_4, $\varkappa_2$, γ_5, $\varkappa_3$, l_2, characterizing cross effects between thermodynamic forces of different time parities, make no contribution to the production entropy, so that the signs of these coefficients remain indeterminate.

The system of equations (1.1), (1.6), (2.16), and (2.21) describes the motion of a nonisothermal, electrically conducting, polarized, and magnetized liquid with particles subject to internal rotation, in the context of electric and magnetic relaxation effects. This is not a closed system in the unknowns ρ, η, p, v, ω, **p**, **m**, **e**, **h**, and T. To obtain a closed system, use must be made of the equations of state (2.18) and (2.11), which connect the variables ρ, η, p, **e**, **h**, **m**, **p**, and T. For example, if the free energy is assumed to be a linear function [13] of the variables J_1 and J_2, with coefficients $1/2\ (\rho\varepsilon_0\chi)^{-1}$ and $\mu_0(2\rho K)^{-1}$ respectively, the expressions $E^{\varepsilon} = (\varepsilon_0\chi)^{-1}\mathbf{p}$ and $E\eta = K^{-1}\mathbf{m}$ are obtained from (2.11) for the local electric and local magnetic field strength vectors, and these close the system with respect to the variables **e**, **h**, **p**, and **m**. In this case, the definitive equation (2.20), describing the time variation in the magnetization vector, can be regarded as a hydrodynamic generalization of the modified Bloch's equation [7] (with one relaxation time, equal to K/r_1). The definitive equation (2.20) for the polarization vector describes a medium of the same type as Voigt's dielectric medium [8]. If the inertia contribution to the change of polarization is neglected in Voigt's definitive equation [8], Eq. (2.20) can be regarded as the hydrodynamic generalization of Voigt's equation.

Notice that, if the quantities **m**, **p**, $\partial e/\partial t$, $\mathbf{v} \times \varepsilon_0\mathbf{e}$, are neglected in Eqs. (1.1), (1.6), (2.16), and (2.21), the result is a system of equations describing the conductive liquid of Grade's model in the approximation of magnetohydrodynamics [16]. The liquid model in question is characterized by mechanical-electric, thermomechanical, and thermoelectric cross effects.

In the absence of interaction between the electromagnetic field and the material medium, our present model is the same as Grade's nonisothermal model [2]. In the case when the quantities $\mathbf{p} \times \mathbf{v}$, $\mathbf{m} \times \mathbf{v}$, ω, ∇, **v**, **s**, **i**, $\partial\mu_0\mathbf{h}/\partial t$, $\partial\mathbf{e}/\partial t$, $\mathbf{v} \times \varepsilon_0\mathbf{e}$, $\mathbf{v} \times \mu_0\mathbf{h}$ can be neglected and the phenomena of dielectric and paramagnetic relaxation are absent, our model is the same as that of [3].

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